On Arbitrarily Partitionable Graphs*

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Abstract

A graph $G = (V, E)$ is said to be arbitrarily partitionable if for any sequence $(n_1, \ldots, n_k)$ of positive integers, whose sum is equal to $|V|$, we can find k disjoint subsets of V, $V_1, \ldots, V_k$, such that each V_i induces a connected subgraph of size n_i. Some families of graphs are known to be arbitrarily partitionable, but determining whether a given graph is arbitrarily partitionable is, in general, tough. We review the main results which have been found so far about this notion.

1 Introduction

Let $G = (V, E)$ be a simple, finite, connected and undirected graph of order n. A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is said *admissible for G* if it sums up to n. If for such an admissible sequence τ we can find a partition $(V_1, \ldots, V_k)$ of V such that each V_i induces a connected subgraph of order n_i , then τ is said to be *realizable in G* and the partition $(V_1, \ldots, V_k)$ is a *realization of τ in G*. G is said to be *arbitrarily partitionable* (AP for short) if every of its admissible sequences can be realized in it.

The notion of AP graphs is mainly linked to the following computer science problem: let us consider a network of connected computing resources, and suppose that there are k different users needing respectively $n_1, \ldots, n_k$ elements of these. In order to optimize the computing power attributed to each user, a resource must be assigned to exactly one of them, and the set of the resources given to a user must be connected. Thus, the most interesting networks for this problem are these which can be distributed this way to users for any sequence of their needs. According to the definition of arbitrarily partitionabilty given above, these are networks which have an AP graph topology.

Arbitrarily partitionability is a quite recent concept first introduced by Barth et al. [3] in 2002[1]. But the problem of partitioning graphs into connected components had been actually studied several times before. For example, in 1975, at the combinatorial colloquium in Aberdeen, A. Frank raised the following conjecture: given a k-connected graph $G = (V, E)$, k positive integers $n_1, \ldots, n_k$ adding up to $|V|$, and k distinct vertices $v_1, \ldots, v_k$ of G, can V be always partitioned into k subsets $V_1, \ldots, V_k$ such that each V_i is of size n_i and contains the vertex v_i? This conjecture, which has been proved independently by Lovász [6] and Győri [4], is quite similar to the notion of arbitrarily partitionability.

Apart from the network context presented above, arbitrarily partitionability revealed itself to be very close-related to some other notions of graph theory. For example, in an AP graph G the sequence $(2, \ldots, 2)$ (resp. $(1, 2, \ldots, 2)$ if its order is odd) can always be realized, and its easy to see that any realization of it in G forms a perfect matching (resp. a quasi-perfect matching). It follows that all AP graphs possess a perfect matching (resp. a quasi-perfect matching). Another noteworthy example is based of the fact that a path is always AP, and thus that a traceable[2] graph is always AP too. Hence, arbitrarily partitionability can be considered as a generalization of hamiltonism [7], a field we still do not know much about.

Determining whether a graph is AP is tough, even for the family of trees: to prove that a graph G is AP, it must be shown that every of its admissible sequences is realizable in it. This becomes

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[1]Under the name of *decomposability* (or *arbitrarily vertex decomposability*), recently changed to avoid any confusion with some other fields of graph theory.

[2]A *traceable graph* is a graph having an hamiltonian path.

quickly unhandleable: not only the problem of determining whether a sequence is realizable in a graph is NP-complete [10], but also the number of such sequences to check is exponential[3].

We review the main results of the investigations which have been carried out so far, whose goals were, mainly, both to find some structural conditions implying the arbitrarily partitionability of graphs and to provide some techniques to easily product a large number of AP graphs.

2 Constrained versions of arbitrarily partitionability

The definition of arbitrarily partitionability presented above is quite lax and does not take heed of the difficulties we can encounter when trying to distribute a given network to some users. Hence, some constrained versions of this notion have been introduced [8] in order to be more representative of these aspects.

2.1 On-line version

One thing which makes a sequence τ easier to realize in a graph G, is that it is entirely known beforehand. Indeed, in this situation, one can analyse the topology of G and thus try to guess where some parts of τ must be placed. This does not appear natural considering the network context presented before, in which the different needs can arrive at different moments.

Hence, we define a new version of AP graphs based on the fact that the parts of a sequence must be placed consecutively, following a certain order. An *on-line arbitrarily partitionable graph* (OL-AP for short) is a graph in which we can realize any of its admissible sequences by placing its parts following their order. Obviously, a graph $G = (V, E)$ is OL-AP if for any integer j between 1 and $|V|$ we can find a subset of V, V_j, which induces a connected subgraph of size j such that the rest of the graph, induced by the vertices of $V - V_j$, is OL-AP.

OL-AP trees have been fully characterized: a tree T is OL-AP if and only if it is a path or a 3-pode[4] having its arms of certain lengths. A class of non-tree and non-traceable graphs has also been characterized: a sun[5] is OL-AP if and only if it has less than five rays, and if its values belong to some determined ones.

2.2 Recursive versions

The definition of AP graphs given in section 1 is quite static, especially when we consider the context of subnetworks attribution. When we perform the distribution of a network, we only make sure that the given resources satisfy the constraints of size and connexity. But what if a user wants, for some reasons, to subdivide his resources again? This will only be possible if the subnetwork we assigned him has an AP graph topology itself.

From this point, a recursive version of the arbitrarily partitionability has been studied: a graph is *recursively arbitrarily partitionable* (R-AP for short) if it is AP with the particularity that any of its admissible sequences can be realized in such a way that the parts induce R-AP subgraphs.

R-AP graphs are quite similar to OL-AP graphs (see section 2.4); hence, trees and suns which are R-AP are nearly the same as the ones which are OL-AP. A class of 2-connected graphs, named balloons[6], has also been studied: R-AP balloons were first thought to have at most six branches [8], but it has been recently shown that there exists some R-AP balloons with an arbitrary number of branches [1].

Note that a stronger definition of the recursively arbitrarily partitionability has been defined: a graph is *strongly recursively arbitrarily partitionable* (SR-AP for short) if it is R-AP with the particularity that every realization of its admissible sequences implies SR-AP subgraphs. SR-AP graphs have been well

[3] The number of different partitions of an integer n is of order $\Omega(e^{\sqrt{n}})$.

[4] A *k-pode* $\mathcal{P}(a_1, \ldots, a_k)$ is a tree consisting of a primary node connected to k disjoint paths, called *arms*, whose orders are respectively $a_1, \ldots, a_k$.

[5] A *sun* $\mathcal{S} = (r_1, \ldots, r_k)$ with k *rays* is a cycle of $n = k + \sum_{i=1}^{k} r_i$ vertices $v_1, \ldots, v_n$ having its nodes v_j connected to a degree one vertex, where $j = 1 + l + \sum_{i=1}^{l} r_i$ for every l between 1 and k.

[6] A balloon $\mathcal{B} = (b_1, \ldots, b_k)$ of k branches is a graph having two distinct vertices linked by k disjoint paths of size b_1, $\ldots$, b_k.

characterized: a graph is SR-AP if and only if it does not contain two particular patterns as induced subgraph: the claw[7] and the net[8].

2.3 With imposed vertices version

In the description of AP graphs, there are no constraints on the parts in which the vertices of the graphs must be placed: we are free to put the vertices in the parts we want as soon as the resulting realization is correct. But, we could like, for some reasons, to have some given vertices in some given parts. This could be the case when, in the network context, a computing resource is the only one having a particular function. Thus, a user could absolutely want to have it in his own subnetwork.

Directly resulting from this problem, a graph G is said to be *arbitrarily partitionable with k imposed vertices* (AP+k for short) if for any sequence $\tau = (n_1, \ldots, n_m)$ admissible for G and k of its nodes $v_1, \ldots, v_k$, associated to an integer between 1 and m by a function ρ, we can find a realization $V_1, \ldots, V_m$ of τ in G in such a way that each v_i lies in the part $V_{\rho(v_i)}$.

This version of arbitrarily partitionability has not been introduced in any publication yet, but fixing some vertices into some parts appears to be a good way to simplify some proofs.

2.4 AP versions hierarchy

It has been shown that all these definitions induce the following hierarchy :

$$\text{PM}^9 \supsetneq \text{AP} \supsetneq \text{OL-AP} \supsetneq \text{R-AP} \supsetneq \text{Traceable} \supsetneq \text{SR-AP}$$

Thus, all these versions become quite useful to prove that a given graph G is AP: if one can show that G is "more" than AP (i.e. AP with some more constraints), then it is obviously AP. Note that the place of the brand new AP+k version in this hierarchy is still unknown, even for $k = 1$.

3 Arbitrarily partitionability of trees

An obvious reason for studying the arbitrarily partitionability of trees is that the absence of cycles in their structure restricts the ways of realizing sequences in them. But what is more interesting is that the property of being AP is closed under edge addition: graphs resulting from some edge additions in an AP graph are AP too. Said differently, if a graph has an AP spanner[10], then so it is. As a result, adding edges in AP trees seemed to be an easy method for generating far more AP graphs.

The majority of the investigations on AP trees intended to characterize their maximum degree Δ, parameter which appeared to be the most natural to consider at first glance. It looks evident that any tree T with $\Delta(T) \leq 2$ is a path, and thus is AP. There also exists some trees which are AP and of maximum degree three: one can easily check that the 3-pode $\mathcal{T}(1, 1, 2)$ is AP. Therefore, the maximum degree of AP trees is at least three.

In 2003, Horňák and Woźniak proved that AP trees are of maximum degree at most six [5]. The main problem of their proof is its non-constructiveness; in particular, it does not assure the existence of AP trees of maximum degree four, five or six.

This bound has been improved in 2006 by Barth and Fournier, who showed that the maximum degree of AP trees is actually four, and that this bound is tight [2]. Their proof is based on the fact that, given an AP tree T and one of its node v of degree d, the d-pode $\mathcal{D}(a_1, \ldots, a_d)$, where the a_i are the sizes of the d components resulting from the deletion of v in T, is also AP. It is then proved that k-podes can not be arbitrarily partitionable when $k > 4$, and, as a consequence, that AP trees can not have nodes of degree more than four.

During their proof, Barth and Fournier found a constraint for AP trees for being of maximum degree four: their degree four nodes must be adjacent to a leaf. Hence, "complex" AP trees are mainly composed

[7] A *claw* is a graph isomorphic to $K_{1,3}$.
[8] A *net* is a triangle K_3 whose vertices are each connected to a pendant vertex.
[9] *PM* is the set of graphs which have a perfect (or quasi-perfect) matching.
[10] A *spanner* of a graph G is a spanning subgraph of G.

of degree three nodes. The number of such nodes in AP trees is not bounded: [2] provides a technique for building an AP $(\alpha+1)$-comb[11] by extending an AP α-comb.

4 Open questions and perspectives

As arbitrarily partitionability is a rather recent concept, the studies done so far mainly aimed to give a first global approach of it; that is why they were essentially focusing on the family of trees. Some more complex structures are hence expected to be investigated: for instance, all the results known about the different versions presented in section 2 principally concern trees. It now appears natural to extend our researches to some more complicated structures to improve our knowledge of arbitrarily partitionability.

One also has to keep in mind that the work done so far on trees shall be improved as we discover new techniques for proving things about arbitrarily partitionability. For example, the AP+k version recently introduced will allow us to fix some "strategical vertices" into some particular parts of a sequence. Being able to satisfy this need, which has been felt in many proofs, will certainly be helpful to simplify some of these.

We have seen in section 3 that the study of AP trees was justified by the fact that adding them edges is a good method to obtain a large number of other AP graphs. But this construction is not sufficient enough to generate all AP graphs: there exists some AP graphs which do not contain an AP spanning tree [9] [8]. Hence, some non-tree AP graphs can also be used as "starting points" for the construction.

From this, another interesting class of AP graphs is being studied. A graph is said to be *minimal arbitrarily partitionable* (min-AP for short) if it is minimal for the property of being AP: removing any edge from it makes it lose this property. The more min-AP graphs we shall find, the more AP graphs we shall thus be able to product by adding them edges.

Apart from the characterization of min-AP trees, we still do not know much about min-AP graphs. A very rough characterization of their maximum degree has been given by Ravaux [9], who proved that if $G = (V, E)$ is an min-AP graph, then it is of maximum degree at most $|V| - 2$. He also conjectured that their number of edges is linear in their number of nodes, in other words that $|E| = O(|V|)$.

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[11] An *α-comb* is a tree of maximum degree 3 having α degree three nodes located on a same path.